

Homogeneity in the free group

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Abstract

We show that any non abelian free group \mathbb{F} is strongly \aleph_0 -homogeneous, i.e. that finite tuples of elements which satisfy the same first-order properties are in the same orbit under $\text{Aut}(\mathbb{F})$. We give a characterization of elements in finitely generated groups which have the same first-order properties as a primitive element of the free group. We deduce as a consequence that most hyperbolic surface groups are not \aleph_0 -homogeneous.

1 Introduction

Since the works of Sela [Sel01]-[Sel06] and Kharlampovich-Myasnikov [KM06] on Tarski's problem, which show that finitely generated free groups of rank at least 2 all have the same first-order theory, there has been renewed interest in model-theoretic questions about free groups. With Sela's work, it has become clear that techniques of geometric group theory provide extremely effective ways to tackle these questions. This can be seen in subsequent results such as [Selc], where Sela shows that the theory of the free group is stable, or in [Per09], where elementary subgroups of a free group of finite rank are shown to be exactly its free factors. Moreover, the geometric nature of the tools often allows these results to be generalized to the class of torsion-free hyperbolic groups (see [Selb]). In this paper, we apply these techniques to study types of elements and homogeneity in torsion-free hyperbolic groups, and more particularly in free and surface groups.

The type of a tuple \bar{a} in a structure is the set of all first-order formulas it realizes. More formally, let \mathcal{L} be a first-order language, \mathcal{M} an \mathcal{L} -structure, and B a subset of \mathcal{M} (for basic definitions of model theory, the reader is referred to [CK90, Mar02], or to the short survey given in [Cha]). A k -type over B is a consistent set of first-order formulas $\phi(\bar{x}) = \phi(x_1, \dots, x_k)$ with k free variables and parameters from B . We say that a k -type $p(\bar{x})$ over B is complete if for every formula $\phi(\bar{x})$ over B , either $\phi(\bar{x})$ or $\neg\phi(\bar{x})$ is in $p(\bar{x})$. Now if \bar{a} is a k -tuple of elements of \mathcal{M} , the set $\text{tp}^{\mathcal{M}}(\bar{a}/B)$ of all formulas with k free variables and parameters in B that \bar{a} realizes in \mathcal{M} is a complete k -type $p(\bar{x})$. However, a complete type need not be of this form (we then say the type is not realized in \mathcal{M}). From now on by a type we mean a complete type.

An \mathcal{L} -structure \mathcal{M} is strongly \aleph_0 -homogeneous if for any pair of finite tuples \bar{a}, \bar{a}' with $\text{tp}^{\mathcal{M}}(\bar{a}) = \text{tp}^{\mathcal{M}}(\bar{a}')$, there is an automorphism of \mathcal{M} sending \bar{a} to \bar{a}' . For more general definitions, see Section 2.1. In the rest of this paper, the structures we will consider are groups and $\mathcal{L} = \{\cdot, ^{-1}, 1\}$ is the language of groups.

Nies proved in [Nie03] that the free group in two generators \mathbb{F}_2 is strongly \aleph_0 -homogeneous, but the question remained opened for free groups of higher rank. As for many reasons \mathbb{F}_2 is a special case, it was clear that the techniques used in [Nie03] do not apply to higher rank free groups. The main result of this paper is

Theorem 1.1: *Non abelian free groups are strongly \aleph_0 -homogeneous.*

In fact, for free groups of finite rank we prove a stronger result: we show that if two tuples have the same type over a set of parameters B , there is an automorphism which sends one to the other and fixes B pointwise.

Note that in [Sk1], it was shown that free groups of uncountable rank are not \aleph_1 -homogeneous, so our result completes the picture of the homogeneity spectrum of the free groups.

In a stable group, there is a distinguished kind of types called the generic types (see Section 2). Following Sela's result on the stability of the theory of finitely generated free groups, Pillay showed in [Pil08] that the theory of the free group admits a unique generic type p_0 , and in [Pil09] that the elements realizing it are exactly the primitive elements (i.e. elements which belong to a basis). The type of primitive elements is thus of particular interest to model theorists, and much work has been done towards understanding it (see [Pil08], [Pil09], [Sk1]).

We generalize Pillay's result by giving a characterization of elements of a finitely generated group G whose type is p_0 in G (note that if such elements exist G must be elementarily equivalent to the free group).

Proposition 1.2: *Let G be a finitely generated group. Let u be an element of G . The type of u in G is p_0 if and only if G admits a structure of hyperbolic tower over $\langle u \rangle$.*

Hyperbolic towers are structures defined by Sela, they are fundamental groups of complexes obtained by successive gluing of hyperbolic surfaces along their boundary to a ground floor complex, in such a way that the surfaces retract on the lower levels. For a formal definition, see Section 6.

An immediate consequence of this characterization is that p_0 is not realized in the fundamental group of the connected sum of four projective planes. In other surface groups, we show using this characterization that the type p_0 is realized both by elements which represent simple closed curves on the surface and elements which don't. An immediate consequence is the following:

Proposition 1.3: *Let Σ be a closed hyperbolic surface which is not the connected sum of three or four projective planes. Then the set of elements of $\pi_1(\Sigma)$ representing simple closed curves on Σ is not definable over the empty set.*

We also get:

Proposition 1.4: *Let Σ be a closed hyperbolic surface which is not the connected sum of three or four projective planes. Then $\pi_1(\Sigma)$ is not strongly \aleph_0 -homogeneous.*

The paper is structured as follows: we start by giving in Section 2 basic definitions and examples about homogeneity, and some overview of stable groups and of the model theoretic properties of the free groups. We devote Section 3 to the description of tools which are essential in proving strong \aleph_0 -homogeneity of the finitely generated free groups: the modular group, and JSJ decompositions. In Section 4, we recall some results about morphisms between torsion-free hyperbolic groups, obtained by the shortening argument of Rips and Sela. We are then able in Section 5 to show Theorem 1.1 in a special case, under some additional assumption on the tuples \bar{a} and \bar{a}' . This example is given as a toy case to help understand how the proof works in general, it is not needed in the proof of the main result. The following section describes the structure of hyperbolic tower, and states results which enable us to claim that a group admits such a structure. In Section 7, we prove Theorem 1.1. The goal of Section 8 is to prove the characterization of elements of type p_0 in a finitely generated group G given by Proposition 1.2.

Finally, in Section 9, we use this characterization to deduce the non homogeneity of surface groups.

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2 Some model theory

The aim of this section is give the definitions of the various notions of homogeneity, as well as a brief overview of some notions which are of interest to place the results of this paper in their model-theoretic context.

2.1 Homogeneity: definitions and examples

We now want to give a formal account of the different notions of homogeneity, as well as some examples. Again we refer the reader to [CK90, Mar02], or to the short survey given in [Cha] for basic model theory definitions.

Definition 2.1: *Let κ be an infinite cardinal. An \mathcal{L} -structure \mathcal{M} is κ -homogeneous if whenever $\beta < \kappa$ and \bar{a}, \bar{b} are β -tuples of elements of \mathcal{M} with $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b})$, and c is an element of \mathcal{M} , then there is d in \mathcal{M} such that $tp^{\mathcal{M}}(\bar{a}c) = tp^{\mathcal{M}}(\bar{b}d)$.*

A slightly stronger notion is that of strong κ -homogeneity.

Definition 2.2: *Let κ be an infinite cardinal. An \mathcal{L} -structure \mathcal{M} is strongly κ -homogeneous if whenever $\beta < \kappa$ and \bar{a}, \bar{b} are β -tuples from \mathcal{M} with $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b})$ then there is an automorphism of the \mathcal{L} -structure \mathcal{M} which sends \bar{a} to \bar{b} .*

In both cases an \mathcal{L} -structure \mathcal{M} is called homogeneous (respectively strongly homogeneous) if it is $|\mathcal{M}|$ -homogeneous (respectively strongly $|\mathcal{M}|$ -homogeneous).

One can easily see that if a structure is strongly κ -homogeneous then it is κ -homogeneous. A less trivial observation is the following

Proposition 2.3: *A structure is homogeneous if and only if it is strongly homogeneous.*

The proof uses a classical back and forth construction.

There are a few results in model theory connecting homogeneity with other model theoretic properties. For example, a κ -saturated model is κ -homogeneous. Also, an atomic model (that is a model \mathcal{A} for which $tp^{\mathcal{A}}(\bar{a})$ is isolated for all $\bar{a} \in \mathcal{A}^n$) is \aleph_0 -homogeneous. Along the same line, one can see that all models of an \aleph_0 -categorical theory are \aleph_0 -homogeneous.

The following example is more natural, and also points out the importance of the language considered: indeed, as shown in [NS90], any model of $\mathcal{Th}(\mathbb{Z}, 1)$ is \aleph_0 -homogeneous. But not all models of $\mathcal{Th}(\mathbb{Z})$ are \aleph_0 -homogeneous, so it is essential that 1 be named. More generally, if G is an infinite abelian group of bounded exponent, then $\mathcal{Th}(G)$ is \aleph_0 -categorical (see [Mar02, Corollary 4.4.4]). Hence by the previous remarks, all models of $\mathcal{Th}(G)$ are \aleph_0 -homogeneous.

We finally give an example demonstrating that the notions of strong κ -homogeneity and κ -homogeneity do not coincide in general.

Example 2.4: Let $\mathcal{L} = \{E\}$ be the language consisting of a single binary relation, and let \mathcal{M} be an \mathcal{L} -structure where E is interpreted as an equivalence relation with two equivalence classes, one of cardinality \aleph_0 and the other of cardinality \aleph_1 . Then $\mathcal{Th}(\mathcal{M})$ is \aleph_0 -categorical,

thus \mathcal{M} is \aleph_0 -homogeneous. But if we take $a, b \in \mathcal{M}$ with $\neg E^{\mathcal{M}}(a, b)$, then $tp^{\mathcal{M}}(a) = tp^{\mathcal{M}}(b)$, and it is easy to check that there is no automorphism of \mathcal{M} taking a to b .

2.2 Stable groups

Stability theory is one of the main recent developments of model theory. It was introduced by Shelah for the purpose of distinguishing wild structures from tame ones from the point of view of first-order logic. The stability of a first-order theory can be given a number of equivalent definitions: we give one which states that it is impossible to "encode" an infinite linear order in a model of the theory.

Definition 2.5: *A complete theory T is stable if there do not exist a formula $\delta(x, y)$, a model \mathcal{M} of T , and sequences $(a_i)_{i < \omega}$ and $(b_i)_{i \in \omega}$ of elements of \mathcal{M} such that $\mathcal{M} \models \delta(a_i, b_j)$ if and only if $i < j$.*

Structures whose first-order theory is stable are in some sense well-behaved, whereas it is often very difficult to say anything about unstable theories. For an introduction to stability theory, the reader is referred to [Pil96].

Within this framework, stable groups hold a special position. There is an independent theory developed for them including concepts such as generic types and connected components (see [Poi01] or turn to the introduction of [Pil08]). For the benefit of the reader we will now explain a few model theoretic notions about groups, with some emphasis on stable groups.

In model theory, a group (G, \cdot) is a structure equipped with a group operation, but possibly also with some additional relations and functions. Even when we do not have explicitly any additional relations or functions, all the sets $X \subseteq G^n$ which are definable over the group under consideration will be part of our structure.

We define a stable group to be a group definable in a stable theory. By this we mean that (G, \cdot) is definable in a model \mathcal{M} of the stable theory T , and it may be equipped with some or all of the structure induced from \mathcal{M} . A typical example is when G is an algebraic group over an algebraically closed field, and we think of G as equipped with predicates for all Zariski closed subsets of G^n .

The simplest case is of course when the group coincides with the ambient structure, and indeed this is the case for non abelian free groups.

Definition 2.6: *Let G be a group. Let X be a definable subset of G . We say that X is left-generic (right-generic) if finitely many left (right) translates of X by elements of G cover G .*

Note that for a definable subset X of a stable group G , it can be shown that X is left generic iff X is right generic, so for stable groups we will simply say generic. We will also say that a formula $\phi(x)$ with one free variable is generic in G if the set $X = \{g \in G \mid G \models \phi(g)\}$ it defines is generic.

Definition 2.7: *Let G be a group. We say that G is connected if it has no definable proper subgroup of finite index.*

Definition 2.8: *Let G be a stable group. Let g be an element of G , and let A be a set of parameters from G . We say that $tp^G(g/A)$ is a generic type if every formula in $tp^G(g/A)$ is generic.*

We have the following useful fact:

Fact 2.9: *Let G be a stable group. Then G is connected if and only if there is over any set of parameters a unique generic type.*

2.3 Free groups

Let κ, λ be cardinals, and let \mathbb{F}_κ denote the free group of rank κ . If $\kappa \leq \lambda$, any injection of κ into λ induces an embedding of \mathbb{F}_κ into \mathbb{F}_λ as a free factor.

We say that an element of \mathbb{F}_κ is primitive if it is part of some basis of \mathbb{F}_κ , analogously, a set of elements of \mathbb{F}_κ is primitive if it can be extended to some basis of \mathbb{F}_κ . As mentioned above, Sela proved in [Sel06] that finitely generated free groups of rank at least 2 have the same first-order theory. In fact he showed the following

Theorem 2.10: *Let $m, n \in \mathbb{N}$ with $2 \leq m \leq n$. Any embedding of \mathbb{F}_m into \mathbb{F}_n as a free factor is elementary.*

A sequence of \mathcal{L} -structures $(\mathcal{M}_i)_{i \in I}$ is an elementary chain if for any i, j with $i \leq j$, the structure \mathcal{M}_i is an elementary substructure of \mathcal{M}_j . It is easy to see that the union $\mathcal{M}_I = \bigcup_{i \in I} \mathcal{M}_i$ can be seen as an \mathcal{L} -structure, and it is a classical result of model theory that \mathcal{M}_i is an elementary substructure of \mathcal{M}_I for each $i \in I$ (see [Mar02]).

Theorem 2.10 shows that $(\mathbb{F}_n)_{n \in \omega}$ with the canonical embedding of \mathbb{F}_m in \mathbb{F}_n as a free factor is an elementary chain. Now clearly the union of $(\mathbb{F}_n)_{n \in \omega}$ is isomorphic to \mathbb{F}_{\aleph_0} : we see that for $n \geq 2$, any embedding of \mathbb{F}_n as a free factor in \mathbb{F}_{\aleph_0} is elementary. In particular, \mathbb{F}_{\aleph_0} is elementary equivalent to \mathbb{F}_n .

It is straightforward to see that Theorem 2.10 also holds for free groups of infinite rank, using for example the Tarski-Vaught test (see [Mar02]):

Theorem 2.11 (Tarski-Vaught Test): *Suppose that \mathcal{M} is a substructure of \mathcal{N} . Then \mathcal{M} is an elementary substructure of \mathcal{N} iff for any formula $\phi(x, \bar{y})$ and $\bar{a} \in \mathcal{M}$, if there is $b \in \mathcal{N}$ such that $\mathcal{N} \models \phi(b, \bar{a})$ then there is $c \in \mathcal{M}$ such that $\mathcal{N} \models \phi(c, \bar{a})$.*

Lemma 2.12: *Let κ, λ be infinite cardinals, with $\kappa \leq \lambda$. Then any embedding of \mathbb{F}_κ in \mathbb{F}_λ as a free factor is elementary.*

Proof. Let $\phi(x, \bar{y})$ be an \mathcal{L} -formula, let $\bar{a} \in \mathbb{F}_\kappa$, and suppose there exists $b \in \mathbb{F}_\lambda$ such that $\mathbb{F}_\lambda \models \phi(b, \bar{a})$. By the Tarski-Vaught test, we only need to show that there is $c \in \mathbb{F}_\kappa$ such that $\mathbb{F}_\lambda \models \phi(c, \bar{a})$. This will be an immediate consequence of the following easy claim.

Claim: For any finite primitive set P of \mathbb{F}_κ and any element b of \mathbb{F}_λ , there is an automorphism f of \mathbb{F}_λ fixing P pointwise such that $f(b) \in \mathbb{F}_\kappa$.

Now, the elements of the tuple \bar{a} are words over some finite primitive set P of \mathbb{F}_κ since by hypothesis $\bar{a} \in \mathbb{F}_\kappa$. By the claim above, there is an automorphism of \mathbb{F}_λ which fixes P pointwise and such that $f(b) \in \mathbb{F}_\kappa$. Thus $\mathbb{F}_\lambda \models \phi(b, \bar{a})$ iff $\mathbb{F}_\lambda \models \phi(f(b), \bar{a})$, and this finishes the proof. \square

From now on we denote by T_{fg} the common theory of the non abelian free groups. The following astonishing result has been proved by Sela in [Selc].

Theorem 2.13: *T_{fg} is stable.*

The above theorem clearly introduces new tools in the studying of free groups. In [Pil08], Pillay observed that the free group \mathbb{F}_{\aleph_0} is connected. As connectedness is a first order property, this shows that any model of T_{fg} is a connected stable group.

By Fact 2.9, connectedness of a stable group implies that there is a unique generic type over any set of parameters. We denote by p_0 the unique generic type of T_{fg} over \emptyset .

Towards the understanding of p_0 , Pillay proved in [Pil09] the following:

Theorem 2.14: *Let a be an element of \mathbb{F}_n with $n \geq 2$. If a realizes p_0 in \mathbb{F}_n , then a is primitive.*

So the primitives of \mathbb{F}_n are exactly the elements which realize p_0 in \mathbb{F}_n . The proof uses essentially the characterization given in [Per09] of elementary embeddings in finitely generated free groups. Some more results concerning p_0 are proved in [Pil08, Pil09, Skl].

3 Modular group and JSJ decomposition

Let G be a finitely generated group. Let Λ be a one-edge cyclic splitting of G , that is, a decomposition of G as an amalgamated product $A *_C B$ or as an HNN extension $A *_C$ over an infinite cyclic subgroup C of G . Recall that a subgroup H of G is said to be elliptic in Λ if it is contained in a conjugate of A or B (respectively in a conjugate of A for the case of an HNN extension).

Definition 3.1: *The Dehn twist of Λ by an element γ of the center of C is the automorphism which restricts to the identity on A and to conjugation by γ on B (respectively the automorphism of G which restricts to the identity on A and sends the stable letter t to $t\gamma$).*

In this section we will often assume the group G to be torsion-free hyperbolic, though the definitions and results we give extend to more general cases. However, we will only use them in this specific setting which simplifies both the definitions and the results.

Definition 3.2: *Let G be a freely indecomposable torsion-free hyperbolic group. The (cyclic) modular group $\text{Mod}(G)$ of G is the subgroup of $\text{Aut}(G)$ generated by Dehn twists of one-edge cyclic splittings of G .*

Let G be a torsion-free hyperbolic group which is freely indecomposable with respect to some subgroup H . The (cyclic) modular group $\text{Mod}_H(G)$ of G relative to H is the subgroup of $\text{Aut}_H(G)$ generated by the Dehn twists fixing H of the one-edge cyclic splittings of G in which the subgroup H is elliptic.

It is a results of Rips and Sela (see [RS94]) that in the torsion-free hyperbolic case, the modular group has finite index in the group of automorphisms.

The idea of the JSJ decomposition is to encode all possible splittings of G over a given class \mathcal{A} of subgroups in a single graph of group Γ (for definitions and results about graphs of groups see [Ser83]). Various results proving the existence of a JSJ decomposition and describing its properties for different hypotheses on G and \mathcal{A} are obtained in [RS97, FP06, DS99, Bow98]. We use the unifying framework of Guirardel and Levitt, developed in [GL09a, GL09b]. We now summarize briefly the definitions and results we will use.

Given a class \mathcal{A} of subgroups of G which is stable under conjugation and taking subgroups, we consider the class of all \mathcal{A} -trees, namely all the simplicial trees endowed with an action of G whose edge stabilizers are in the class \mathcal{A} . One might also be interested in the JSJ relative to some subgroup H of G : in this case, one considers only the \mathcal{A} -trees in which H is elliptic (that is, fixes a vertex). We will call such trees (\mathcal{A}, H) -trees.

Given \mathcal{A} -trees T and T' , we say that T **dominates** T' if there exists a G -equivariant continuous map $T \rightarrow T'$, and that T **refines** T' (or **collapses to** T') if this map consists in

collapsing some of the edges of T to vertices. The **deformation space** of an \mathcal{A} -tree T is the set of all \mathcal{A} -trees T' such that T dominates T' and T' dominates T . An \mathcal{A} -tree is **universally elliptic** if its edge stabilizers are elliptic in every \mathcal{A} -tree. If T is a universally elliptic \mathcal{A} -tree, and T' is any \mathcal{A} -tree, it is easy to see that there is a tree \hat{T} which refines T and dominates T' (see [GL09a, Lemma 3.2]).

A **JSJ tree** is a universally elliptic \mathcal{A} -tree which dominates any other universally elliptic tree. All JSJ-trees belong to a same deformation space, that we denote \mathcal{D}_{JSJ} . Guirardel and Levitt show that if G is finitely presented, the JSJ deformation space always exists, without any restrictions on \mathcal{A} (see [GL09a, Theorem 4.2]). This extends to the relative case (replacing 'every \mathcal{A} -tree' by 'every (\mathcal{A}, H) -tree' where appropriate in the definitions), provided H is also finitely generated.

The real heart of the theory of JSJ decompositions is to describe the properties of the JSJ trees, and sometimes to find a canonical tree in \mathcal{D}_{JSJ} . A vertex stabilizer in a (relative) JSJ tree is said to be **rigid** if it is elliptic in any \mathcal{A} -tree (respectively any (\mathcal{A}, H) -tree), and **flexible** if not. Interpreting results of [RS97, DS99, FP06] in this framework gives a description of flexible vertices under some conditions on \mathcal{A} (see [GL09a, Theorem 7.7] and [GL09a, Theorem 7.38] for the relative case). We will give this description in the special case where G is torsion-free hyperbolic and freely indecomposable (respectively freely indecomposable with respect to the finitely generated subgroup H). We first give

Definition 3.3: *Fix a group G . Let T be a tree endowed with an action of G (respectively an action of G in which H is elliptic). We say that a vertex v of T is of surface type if*

- *the stabilizer of v is the fundamental group S of a hyperbolic surface with boundary Σ ;*
- *each incident edge stabilizer is contained in a boundary subgroup of S ;*
- *each maximal boundary subgroup of S contains as a subgroup of finite index the stabilizer of an incident edge (or a conjugate of H in the relative case).*

If Λ is the graph of group corresponding to T , we also say that the vertex of Λ corresponding to v is of surface type.

We now have:

Theorem 3.4: *Let G be torsion-free hyperbolic and freely indecomposable (respectively freely indecomposable with respect to a non trivial finitely generated subgroup H). Let T be a (relative) JSJ tree over the class \mathcal{A} of cyclic subgroups. Then the flexible vertices of T are of surface type.*

The first property of JSJ trees we will use is that its vertex groups are preserved in some sense by modular automorphisms.

Lemma 3.5: *Let G be torsion-free hyperbolic and freely indecomposable (respectively freely indecomposable with respect to a non trivial finitely generated subgroup H). Let T be a (relative) JSJ tree over the class \mathcal{A} of cyclic subgroups. An element of $\text{Mod}(G)$ (respectively $\text{Mod}_H(G)$) restricts to conjugation on each rigid vertex stabilizer of T , and sends flexible vertex stabilizers isomorphically on conjugates of themselves.*

Proof. Let T_0 be the \mathcal{A} -tree corresponding to a one-edge cyclic splitting of G along a cyclic subgroup C , and let τ be the Dehn twist corresponding to an element γ of C . Let v be a

vertex of T with vertex group V . If v is elliptic in T_0 (in particular if v is rigid), $\tau|_V$ is just a conjugation.

Suppose now that V is not elliptic in T_0 : in particular, v must be flexible, so it is of surface type. We will show that (some conjugate of) γ represents a simple closed curve on the corresponding surface Σ . The boundary subgroups of V are elliptic in any \mathcal{A} -tree since T is universally elliptic. Thus by Proposition III.2.6 of [MS84], V inherits from its action on T_0 a splitting which is dual to a set of essential simple closed curves on the surface corresponding to V . Thus, up to conjugation, γ is the root of an element c of V which represents an essential simple closed curve on Σ . Now the orbit of the vertex v under $\langle \gamma \rangle$ is finite, so γ must fix a vertex w of T . If $v \neq w$, the path between v and w is fixed by c , so c is a boundary element of V . But this contradicts essentiality of the simple closed curve it represents. Hence γ fixes v , so it lies in V and represents a simple closed curve on the corresponding surface. The Dehn twist τ by γ is thus an automorphism of V , so τ sends V isomorphically to a conjugate of itself. \square

We will pick among the trees in \mathcal{D}_{JSJ} a particular JSJ tree with specific properties: the tree of cylinders. It turns out to be precisely the JSJ decomposition given by Bowditch in [Bow98]. In our setting, the cylinders of an \mathcal{A} -tree T (respectively of an (\mathcal{A}, H) -tree) are the equivalence classes of edges under the relation given by commensurability of stabilizers. Cylinders are subtrees of T , and the tree of cylinders T_C of T (which can be shown to depend only on the deformation space of T) is built as follows: its set of vertices is the union of the set $C(T)$ of cylinders of T and the set $\partial C(T)$ of vertices of T which lie in two distinct cylinders, and there is an edge between a vertex w of $C(T)$ and a vertex v of $\partial C(T)$ if v lies in the cylinder corresponding to w . The following properties of T_C are given in [GL08, Theorem 2].

Theorem 3.6: *Let G be torsion-free hyperbolic and freely indecomposable (respectively freely indecomposable with respect to a non trivial finitely generated subgroup H). Let \mathcal{A} be the class of cyclic subgroups of G . Let T_C be the tree of cylinders of an \mathcal{A} -tree T . Then T_C is itself an \mathcal{A} -tree which belongs to the same deformation space as T , in particular if T is a JSJ tree, so is T_C . Moreover, T_C is 2-acylindrical.*

From now on, we will call any decomposition of a group G corresponding to a (relative) JSJ tree **a** (relative) JSJ decomposition of G , but **the** (relative) JSJ decomposition of a group G will refer to the decomposition corresponding to the tree of cylinders of the (relative) JSJ deformation space.

Remark 3.7: *It is straightforward to see by construction of the tree of cylinders that in the tree of cylinders of the JSJ deformation space, all the stabilizers of edges incident to a flexible vertex have index 1 in the corresponding boundary subgroup, and two incident edges whose stabilizers correspond to conjugate boundary subgroups are in the same orbit.*

Finally, we will need the following result.

Proposition 3.8: *Let G be a torsion-free hyperbolic group. Let H be a subgroup of G with respect to which G is freely indecomposable. There exists a finitely generated subgroup H_0 of H such that G is freely indecomposable with respect to H_0 , and H is elliptic in the JSJ decomposition of G relative to H_0 . Moreover, we have $\text{Mod}_{H_0}(G) = \text{Mod}_H(G)$.*

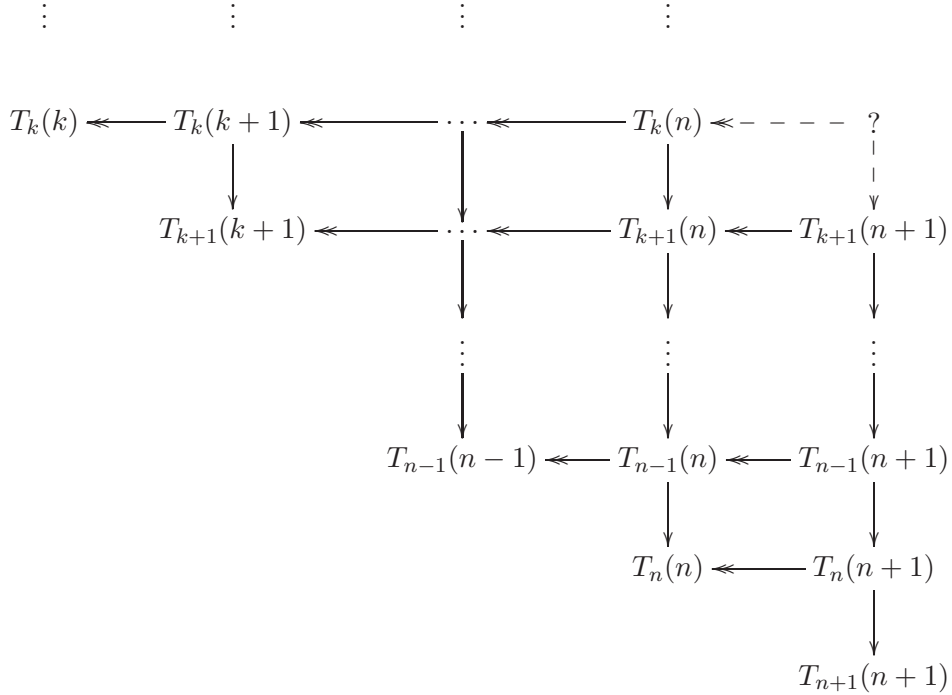
Proof. Consider an exhausting chain of finitely generated subgroups H_n of H . By Lemma 4.23 in [Per09] (see also Lemma 6.6 of [GL09b]), we may assume that G is freely indecomposable with respect to all the subgroups H_n .

For each n , we let T_n be a tree in the JSJ deformation space of G relative to H_n . The tree T_n does not necessarily dominate T_m for $m \geq n$, but we claim that there exists for each n a refinement \hat{T}_n of T_n which dominates \hat{T}_m for all $m \geq n$.

Suppose we do have such a sequence \hat{T}_n . Let S_n be the tree of cylinders of \hat{T}_n : it is in the same deformation space as \hat{T}_n . By functoriality of the trees of cylinders (see [GL08, Proposition 4.11]), there is a G -equivariant cell map $f_n : S_n \rightarrow S_{n+1}$. This shows that the number of orbits of edges of S_n is non increasing, thus we may assume it is constant. If f_n is not an isomorphism, there must be some foldings, hence some edges whose stabilizer is not maximal cyclic in the adjacent vertex stabilizers. Each folding increases strictly the index of an edge stabilizer in its maximal cyclic subgroup (recall we are in a torsion-free hyperbolic group): this can only happen finitely many times. The sequence S_n eventually stabilizes, so there exists n_0 such that H_n is elliptic in S_{n_0} , and thus in T_{n_0} , for all n large enough.

Finally, the vertex group containing H_{n_0} of the JSJ tree T_{n_0} is elliptic in any one edge cyclic splitting of G in which H_{n_0} is elliptic. This means that H is elliptic in any such splitting, which implies that $\text{Mod}_{H_{n_0}}(G) \subseteq \text{Mod}_H(G)$. The other inclusion is immediate.

There remains only to show that we can build the sequence \hat{T}_n . The idea is the following: we start by refining $T_1(1) = T_1$ to a tree $T_1(2)$ which dominates T_2 . Then, we refine T_2 to a tree $T_2(3)$ which dominates T_3 . We now want to see that we can refine $T_1(2)$ to a tree $T_1(3)$ which dominates also $T_2(3)$.



More formally, we want to build (\mathcal{A}, H_k) -trees $T_k(n)$ for all $n \geq k$ such that

1. $T_k(n)$ is (\mathcal{A}, H_k) -universally elliptic;
2. $T_k(n)$ dominates $T_{k+1}(n)$;
3. $T_k(n)$ is a refinement of $T_k(n-1)$;

(see the commutative diagram, where double arrows represent collapse).

We proceed by (upwards) induction on n . We let $T_1(1) = 1$. Suppose we have built the trees $T_k(j)$ for $k \leq j \leq n$. We now want to build the trees $T_k(n+1)$ for $k \leq n+1$. We start by letting $T_{n+1}(n+1) = T_{n+1}$: it satisfies 1. Now, we proceed by (downwards) induction on k : suppose we have built the trees $T_j(n+1)$ for $n+1 \geq j \geq k+1$. The tree $T_k(n)$ is (\mathcal{A}, H_k) -universally elliptic by induction hypothesis, and $T_{k+1}(n+1)$ is a (\mathcal{A}, H_k) -tree, so we can refine $T_k(n)$ to a tree $T_k(n+1)$ dominating $T_{k+1}(n+1)$. The edge groups of $T_k(n+1)$ which are not already edge groups of $T_k(n)$ are contained in edge groups of $T_{k+1}(n+1)$, which by induction on k are elliptic in any (\mathcal{A}, H_{k+1}) -tree. Thus $T_k(n+1)$ satisfies condition 1.

For each k , the sequence of refinements $(T_k(j))_{j \in \mathbb{N}}$ eventually stabilizes (by Dunwoody accessibility, see for example Proposition 4.3 of [GL09a]). Thus we get a sequence \hat{T}_n of refinements of the JSJ trees T_n for which \hat{T}_n dominates \hat{T}_{n+1} . \square

4 Factor sets and test elements

The following results, whose proofs are all based on the shortening argument, describe various properties of homomorphisms $G \rightarrow \Gamma$ between torsion-free hyperbolic groups, and will therefore be crucial in understanding homogeneity properties of such groups. They are all variations on results proved in [Sel01] for the free case and in [Selb] for the general torsion-free hyperbolic case. Note that most of these results are valid if we only assume G to be finitely generated, but this would require a more general definition of the modular group, and some subtle arguments are required to deal with the axial components in the shortening.

The first result deals with embeddings $G \hookrightarrow \Gamma$.

Theorem 4.1: *Let G and Γ be torsion-free hyperbolic groups. Let H be a subgroup of G with respect to which G is freely indecomposable, and let $f : H \rightarrow \Gamma$ be an embedding of H into Γ .*

Then there exists a finite set $\{i_j : G \rightarrow \Gamma\}_{1 \leq j \leq s}$ of embeddings of G into Γ such that for any embedding $\theta : G \rightarrow \Gamma$ restricting to f on H , there exists an element σ of $\text{Mod}_H(G)$ and an element γ in the centralizer $Z(f(H))$ of $f(H)$ such that $\theta = \text{Conj}(\gamma) \circ i_j \circ \sigma$ for some j .

For monomorphisms, we get

Theorem 4.2: *Let Γ be a torsion-free hyperbolic group which is freely indecomposable with respect to a non trivial subgroup H . Then any injective homomorphism $\Gamma \rightarrow \Gamma$ which restricts to the identity on H is an isomorphism.*

The result also holds when H is trivial, provided Γ is not 2-ended (see Theorem 4.4 of [Sel97]) but the proof is much easier in the relative case.

Remark 4.3: *If H is finitely generated, Theorem 4.2 says exactly that a finite generating set for H is a test tuple for monomorphisms (see Definition 2.3 of [Nie03]). To show homogeneity of \mathbb{F}_2 , Nies uses the fact that for any tuple \bar{g} in \mathbb{F}_2 , either \mathbb{F}_2 is freely indecomposable with respect to the subgroup $\langle \bar{g} \rangle$ (so \bar{g} is a test tuple), or the elements of \bar{g} are all powers of a same primitive element. The second crucial factor in Nies' proof is that the injectivity of a morphism $\mathbb{F}_2 \rightarrow \mathbb{F}_2$ can be expressed in first-order (it is enough to say that the images of the generators do not commute). Both these properties do not hold in higher rank.*

Finally, non injective morphisms are dealt with by

Theorem 4.4: *Let G and Γ be torsion-free hyperbolic groups. Let H be a subgroup of G with respect to which G is freely indecomposable, and let $f : H \rightarrow \Gamma$ be a homomorphism.*

Then there exists a finite set $\{\eta_i : G \rightarrow M_i\}_{1 \leq i \leq s}$ of proper quotients of G such that for any non injective homomorphism $\theta : G \rightarrow \Gamma$ which restricts to f on H , there exists an element σ of $\text{Mod}_H(G)$ such that $\theta \circ \sigma$ factors through one of the maps η_i .

Before proving these three results, we will deduce from Theorem 4.1 the following corollary:

Corollary 4.5: *Let G and Γ be torsion-free hyperbolic groups. Let H be a subgroup of G with respect to which G is freely indecomposable, and let f be an embedding of H into Γ . There is a finitely generated subgroup H_0 of H such that any embedding $\theta : G \rightarrow \Gamma$ which restricts to f on H_0 restricts to f on H .*

Proof of corollary. We may assume H infinitely generated, hence non abelian. We write H as an increasing union of finitely generated subgroups H_n . For n large enough, by Lemma 3.8 we may assume that G is freely indecomposable with respect to H_n , and that $\text{Mod}_{H_n}(G) = \text{Mod}_H(G)$. Since G is freely indecomposable with respect to H_n , Theorem 4.1 gives a finite set I_{H_n} of injective morphisms $G \rightarrow \Gamma$ which restrict to the identity on H_n . For n large enough H_n is non abelian, so that $Z(f(H_n))$ is trivial. Any injective morphism $\theta : G \rightarrow \Gamma$ which restricts to f on H_n is equal to one of the elements of I_{H_n} after precomposition by an element of $\text{Mod}_{H_n}(G)$. In particular, this applies to the injective morphisms contained in $I_{H_{n+1}}$, so we may assume $I_{H_{n+1}} \subseteq I_{H_n}$ since $\text{Mod}_{H_n}(G) = \text{Mod}_{H_{n+1}}(G)$.

This non increasing chain of finite sets stabilizes so we may assume it is constant. Suppose θ restricts to f on H_n . Then it is the composition of an element of $\text{Mod}_{H_n}(G)$ with an element i of I_{H_n} . But any element of $\text{Mod}_{H_n}(G)$ restricts to the identity on H , and i lies in I_{H_n} for all n , so it also restricts to the identity on H . This proves the result. \square

We will now give outlines of the proofs of Theorems 4.2, 4.4 and 4.1. Let G and Γ be torsion-free hyperbolic groups. Let H be a subgroup of G with respect to which G is freely indecomposable, and fix a morphism $f : H \rightarrow \Gamma$. We fix finite generating sets S and Σ for G and Γ respectively, and we denote by $|\cdot|_\Sigma$ the word length with respect to Σ .

Definition 4.6: *We say that a homomorphism $\theta : G \rightarrow \Gamma$ which restricts to f on H is short with respect to H if for any element τ of $\text{Mod}_H(G)$ and any element γ of the centralizer $Z(f(H))$ of $f(H)$ in Γ , we have:*

$$\max_{g \in S} |\theta(g)|_\Sigma \leq \max_{g \in S} |\gamma \theta(\tau(g)) \gamma^{-1}|_\Sigma.$$

Definition 4.7: *A sequence $(\theta_n)_{n \in \mathbb{N}}$ of group homomorphisms $\theta_n : G_1 \rightarrow G_2$ is said to be stable if for any element g of G_1 , either g lies in $\text{Ker } \theta_n$ for all n large enough, or g lies outside of $\text{Ker } \theta_n$ for all n large enough. The set of elements for which the first alternative holds is called the stable kernel of the sequence θ_n , and is denoted by $\underline{\text{Ker}} \theta_n$.*

Remark 4.8: *If G_1 is countable, any sequence of morphisms $G_1 \rightarrow G_2$ has a stable subsequence.*

Proposition 4.9: *Let G and Γ be torsion-free hyperbolic groups, assume G is not cyclic. Let H be a subgroup of G with respect to which G is freely indecomposable, and let f be a morphism $H \rightarrow \Gamma$. Suppose $(\theta_n)_{n \in \mathbb{N}}$ is a stable sequence of distinct morphisms $G \rightarrow \Gamma$ which restrict to f on H , and are short with respect to H . Then its stable kernel $\underline{\text{Ker}} \theta_n$ is non trivial.*

The proof of this is an instance of Rips and Sela's classical shortening argument developed in [RS94] (see [Wil06] or [Per08] for a more detailed exposition), thus we only outline it and detail the arguments which are not classical.

Outline of the proof. Let us assume by contradiction that the stable kernel of the sequence $(\theta_n)_{n \in \mathbb{N}}$ is trivial (this implies in particular that f is injective). Denote by X the Cayley graph of Γ with respect to Σ . The group G acts on X via θ_n . The displacement function $\Delta_n : X \rightarrow \mathbb{N}$ associated to this action is given by

$$\Delta_n(x) = \max_{g \in S} d_X(x, \theta_n(g) \cdot x) = \max_{g \in S} |x^{-1} \theta_n(g) x|_\Sigma.$$

Let μ_n denote the minimum reached by this function on $Z(f(H))$ (where $Z(f(H))$ is seen as a set of vertices in X). Note that the shortness of the maps θ_n implies that this minimum is reached for $x = 1_n$ the vertex of X representing the identity element. We thus have $\mu_n = \max_{g \in S} |\theta_n(g)|_\Sigma$. The maps θ_n are pairwise distinct so the sequence μ_n tends to infinity.

We denote by $X[\theta_n]$ the space X endowed with the metric $d_n = d_X/\mu_n$, with the action of G via θ_n , and with basepoint the vertex 1_n corresponding to the identity element.

After extraction, the sequence $X[\theta_n]$ converges to a pointed real G -tree (T, x) , that we can assume minimal. The triviality of the stable kernel of $(\theta_n)_{n \in \mathbb{N}}$ implies in particular that the image $\theta_n(G)$ is eventually non cyclic, and in this case it can be shown that the action of G on T is faithful, that tripod stabilizers are trivial and that arc stabilizers are abelian. These last two properties imply that the action is superstable.

Let us show that it does not have a global fixed point. By choice of the scaling constant, the basepoint x is not a global fixed point. On the other hand, since the maps θ_n are constant on H and μ_n tends to infinity, H fixes the basepoint. If there is a global fixed point y , then H fixes the arc between x and y , so in particular it is abelian. Since it is a subgroup of the torsion-free hyperbolic group G , it must in fact be cyclic or trivial. Let g be an element of the generating set S of G such that $d(x, g \cdot x)$ is non zero, and pick ϵ much smaller than $d(x, g \cdot x)$. Let now y_n approximate y in $X[\theta_n]$ in an ϵ -approximation with respect to x, y, S and a generator for H . Now for n large enough, we have $d_n(y_n, \theta_n(g)y_n) < d_n(1_n, \theta_n(g)1_n)$, so that $\Delta_n(y_n) < \Delta_n(1_n)$.

If H is trivial, y_n lies in $Z(f(H))$ and this contradicts the fact that 1_n minimizes Δ on $Z(f(H))$. If H is cyclic and generated by h , then for n large enough 1_n and y_n are very close to the axis of h in $X[\theta_n]$. Thus, there exists an integer K_n such that $z_n = f(h)^{K_n}$ is very close to y_n . Now the orbit of y_n by G has small diameter, thus so does the orbit of z_n : this implies that $\Delta_n(z_n) < \Delta_n(1_n)$, which contradicts shortness of θ_n since $z_n = f(h)^{K_n}$ is in $Z(f(H))$. We showed that the action does not admit a global fixed point.

The action of G on T satisfies the hypotheses necessary to be analyzed by Rips theory (see Theorem 5.1 of [Gui08]), this gives us in particular a decomposition Λ of G as a graph of groups with surfaces in which the subgroup H is elliptic. We can now use classical shortening arguments (originally presented in [RS94], see also Theorem 4.28 of [Per08]) to show that we can precompose θ_n by an element σ of the modular group $\text{Mod}(\Lambda)$ in such a way that $\theta_n \circ \sigma$ is shorter than θ_n . This is a contradiction. \square

Proof of Theorem 4.1. By Theorem 4.9, any stable sequence of distinct short morphisms $\theta_n : G \rightarrow \Gamma$ which coincide with f on H has a non-trivial stable kernel. In particular, this means there are only finitely many short injective morphisms $G \rightarrow \Gamma$ restricting to f on H . This proves the result. \square

Proof of Theorem 4.2. Suppose θ is an injective yet non surjective morphism $\Gamma \rightarrow \Gamma$ which restricts to the identity on H .

For each integer n , we consider the n -th power of θ : it is an injective morphism $\theta^n : G \rightarrow G$. We pick an element τ_n of $\text{Mod}_H(G)$, and γ_n of $Z(H)$ such that the morphism $\hat{\theta}_n = \text{Conj}(\gamma_n) \circ \theta^n \circ \tau_n$ is short with respect to H . Note that since H is non trivial and Γ is torsion-free hyperbolic, its centralizer is either trivial, or an infinite cyclic subgroup which contains H as a finite index subgroup. Thus, up to replacing θ by a conjugate, and extracting a subsequence of the θ^n , we may assume that γ_n lies in H and thus in $\theta^n(G)$. In particular we get $\hat{\theta}_n(G) = \gamma_n \theta^n(G) \gamma_n^{-1} = \theta^n(G)$.

The injective morphisms $\hat{\theta}_n : G \rightarrow G$ are pairwise distinct, since their images strictly embed one into the other. Up to further extraction, we may assume that the sequence $(\hat{\theta}_n)_{n \in \mathbb{N}}$ is stable. But Proposition 4.9 for $G = \Gamma$ and f the identity on H tells us that it must then have non trivial stable kernel, which contradicts the fact that the morphisms $\hat{\theta}_n$ are all injective. \square

To prove Proposition 4.4, we need

Proposition 4.10: *Let G be a finitely generated group, and let Γ be a torsion-free hyperbolic group. Suppose \mathcal{S} is a set of stable sequences $(\theta_n)_{n \in \mathbb{N}}$ of homomorphisms $G \rightarrow \Gamma$ which satisfies:*

- \mathcal{S} is closed under extraction of subsequences;
- \mathcal{S} is closed under extraction of diagonal subsequences.

Then there exists a finite number of quotient maps $\eta_j : G \rightarrow G/K_j$ where K_j is the stable kernel of a sequence which lies in \mathcal{S} , such that for any element $(\theta_n)_{n \in \mathbb{N}}$ of \mathcal{S} , the homomorphisms θ_n eventually factor through one of the maps η_j .

This can be proved by following the argument given in [Sel01] to prove Lemma 5.4 and Lemma 5.5 where this is shown in the case where Γ is free (see Proposition 6.21 and 6.22 in [Per08] for a proof in the general torsion-free hyperbolic case).

Proof of Theorem 4.4. Note first that if G is cyclic, any non injective morphism $G \rightarrow \Gamma$ is trivial, since Γ is torsion-free. We can thus assume that G is not cyclic.

Let \mathcal{S} be the set of stable sequences $(\theta_n)_{n \in \mathbb{N}}$ of non injective short morphisms $G \rightarrow \Gamma$ which restrict to f on H . Proposition 4.10 gives us a finite set of quotients of the form $\eta : G \rightarrow G/K$, where K is the stable kernel of a sequence $(\theta_n)_{n \in \mathbb{N}}$ which lies in \mathcal{S} .

Assume K is trivial. We extract from the sequence $(\theta_n)_{n \in \mathbb{N}}$ a subsequence $(\theta_{n_k})_{k \in \mathbb{N}}$ of pairwise distinct morphism as follows: let $n_1 = 1$. Suppose we have picked θ_{n_k} : it is not injective, so we can pick a non trivial element g_k in its kernel. Since the stable kernel of $(\theta_n)_{n \in \mathbb{N}}$ is trivial, there exists $n_{k+1} > n_k$ such that for all $n \geq n_{k+1}$, we have $\theta_n(g_k) \neq 1$. Clearly we get in this way a sequence of pairwise distinct morphisms. By Proposition 4.9, K is non trivial: this is a contradiction.

Let now $\theta : G \rightarrow \Gamma$ be a non injective morphism which restricts to f on H , and let τ in $\text{Mod}_H(G)$ and γ in $Z(f(H))$ be such that $\hat{\theta} = \text{Conj}(\gamma) \circ \theta \circ \tau$ is short with respect to H . The constant sequence $(\hat{\theta})_{n \in \mathbb{N}}$ lies in \mathcal{S} , so $\hat{\theta}$ eventually factors through one of the quotients of our finite set. But this means that $\theta \circ \tau$ factors through this quotient. \square

5 A special case

To give an insight of how we prove homogeneity of finitely generated free groups, we will now show the following result:

Proposition 5.1: *Let \mathbb{F} be a finitely generated group of rank at least 2. Suppose \bar{g}, \bar{g}' are tuples of elements of \mathbb{F} such that $\text{tp}^{\mathbb{F}}(\bar{g}) = \text{tp}^{\mathbb{F}}(\bar{g}')$. Suppose moreover that \mathbb{F} is freely indecomposable with respect to each of the subgroups $H = \langle \bar{g} \rangle$ and $H' = \langle \bar{g}' \rangle$, and that both the JSJ decomposition of \mathbb{F} relative to H and relative to H' are trivial. Then there is an automorphism of \mathbb{F} which sends \bar{g} to \bar{g}' .*

Proof. The triviality of the JSJ decomposition implies that there are no non-trivial cyclic splittings of \mathbb{F} in which H or H' is elliptic, hence $\text{Mod}_H(\mathbb{F})$ and $\text{Mod}_{H'}(\mathbb{F})$ are trivial.

It is enough to show that there exist injective morphisms $i, j : \mathbb{F} \rightarrow \mathbb{F}$ with $i(\bar{g}) = \bar{g}'$ and $j(\bar{g}') = \bar{g}$. Indeed, then $i \circ j$ is an injective automorphism of \mathbb{F} which fixes \bar{g} , so by Theorem 4.2 it is in fact an automorphism. Thus i is also bijective, and we get the result.

Suppose thus by contradiction that (without loss of generality) there is no injective morphism $i : \mathbb{F} \rightarrow \mathbb{F}$ with $i(\bar{g}) = \bar{g}'$. Define the morphism $f : H \rightarrow \mathbb{F}$ by $f(\bar{g}) = \bar{g}'$. We get by Theorem 4.4 a finite set of proper quotients $\{\eta_i : \mathbb{F} \rightarrow M_i\}_{1 \leq i \leq r}$, and since all the morphisms $\mathbb{F} \rightarrow \mathbb{F}$ sending \bar{g} to \bar{g}' are non injective, **any** such morphism factors through one of the maps η_i (we do not need to precompose by anything since the modular group is trivial). We will now translate this statement as a first order sentence satisfied by \bar{g}' over \mathbb{F} , and see that \bar{g} cannot satisfy it: this will contradict \bar{g} and \bar{g}' having the same type.

Fix a basis $\bar{a} = (a_1, \dots, a_n)$ of \mathbb{F} , and write the tuple \bar{g} as a tuple of words $\bar{g}(\bar{a})$ in the basis elements a_i . Pick also a non trivial element v_i in the kernel of each map η_i , and write it as a word $v_i(\bar{a})$. A morphism $\mathbb{F} \rightarrow \mathbb{F}$ is just a choice of image $\bar{x} = (x_1, \dots, x_n)$ for the basis, and the image of the element represented by a word $w(\bar{a})$ is represented by $w(\bar{x})$. Thus the following first-order statement holds on \mathbb{F} :

$$\phi(\bar{g}') : \forall \bar{x} \left\{ [\bar{g}(\bar{x}) = \bar{g}'] \Rightarrow \left[\bigvee_{i=1}^r \bar{v}_i(\bar{x}) = 1 \right] \right\}.$$

The statement $\phi(\bar{g})$ says that any morphism $\mathbb{F} \rightarrow \mathbb{F}$ which sends \bar{g} to \bar{g} kills one of the v_i . But the identity is an obvious counterexample. \square

An interesting exercise is to generalize this proof assuming not that the JSJ decomposition is trivial, but that it has no surface groups (i.e. that all the vertex groups of the JSJ decomposition are rigid). In the general case, however, the modular group does not translate well into first-order, and the difficulty comes precisely from the surface groups.

6 Hyperbolic towers and preretractions

Definition 6.1: *Consider a triple (G, G', r) where G is a group, G' is a subgroup of G , and r is a retraction from G onto G' . We say that (G, G', r) is a hyperbolic floor if there exists a non trivial decomposition Λ of G as a graph of groups with a distinguished subset V_S of vertices which are of surface type such that:*

- *G' is the subgroup of G generated by the groups of the vertices of Λ which are not in V_S , and it is in fact their free product;*
- *the surfaces corresponding to vertices in V_S are either punctured tori or of characteristic at most -2 ;*

- every edge e of Λ has exactly one endpoint v_e in V_S (bipartism), the edge group G_e is a maximal boundary subgroup of the vertex group G_{v_e} , and this induces a bijection between edges of Λ adjacent to v_e and conjugacy classes of maximal boundary subgroups of G_{v_e} ;
- the retraction r sends groups of vertices of V_S to non abelian images.

By an abuse of language, we call the vertices of V_S the surface type vertices of Λ , and other vertices the non surface type vertices (although some of them might have surface type according to Definition 3.3).

Definition 6.2: Let G be a group, let H be a subgroup of G . We say that G is a hyperbolic tower based on H if there exists a finite sequence $G = G^0 \geq G^1 \geq \dots \geq G^m \geq H$ of subgroups of G with $m \geq 0$ and:

- for each k in $[0, m - 1]$, there exists a retraction $r_k : G^k \rightarrow G^{k+1}$ such that the triple (G^k, G^{k+1}, r_k) is a hyperbolic floor, and H is contained in one of the non surface type vertex group of the corresponding hyperbolic floor decomposition;
- G^m is not abelian and $G^m = H * F * S_1 * \dots * S_p$ where F is a (possibly trivial) free group, $p \geq 0$, and each S_i is the fundamental group of a closed surface of Euler characteristic at most -2 .

Remark 6.3: By Bestvina and Feighn's combination theorem (see [BF92]), hyperbolic towers over torsion-free hyperbolic groups are themselves torsion-free hyperbolic groups.

Hyperbolic towers were introduced by Sela in [Sel01], and he shows in [Sel06] the following result:

Theorem 6.4 ([Sel06]): A finitely generated group G is elementary equivalent to a non abelian free group if and only if it has a structure of hyperbolic tower over the trivial subgroup.

Finitely generated free groups admit a structure of hyperbolic tower over any of their free factors. The following result states that this is in fact the only tower structure a free group can admit over one of its subgroups.

Proposition 6.5: There is no retraction r from the free group \mathbb{F}_k to one of its subgroups which makes $(\mathbb{F}_k, r(\mathbb{F}_k), r)$ into a hyperbolic floor.

The proof of this result is contained in the proof of Theorem 1.3 in [Per09].

We now want to give a result which enables us to deduce that a group G has a structure of hyperbolic floor over a proper subgroup, assuming the existence of a map $G \rightarrow G$ which preserves some of the properties of a cyclic (relative) JSJ decomposition for G . We need to define these maps slightly more generally as maps $U \rightarrow G$ where U is a subgroup of G .

Definition 6.6: Let G be a group, let U be a subgroup of G , and let Λ be a cyclic (relative) JSJ decomposition of U . A morphism $U \rightarrow G$ is a preretraction with respect to Λ if its restriction to each rigid vertex group U_v of Λ is a conjugation by some element g_v of G , and if flexible vertex groups have non abelian images.

The following result appears as Proposition 5.11 in [Per09].

Proposition 6.7: Let U be a torsion-free hyperbolic group. Let Λ be a cyclic (relative) JSJ decomposition of U . Assume that there exists a non injective preretraction $U \rightarrow U$ with respect to Λ . Then there exists a subgroup U' of U and a retraction r from U to U' such that (U, U', r)

is a hyperbolic floor. Moreover, given a rigid type vertex group R_0 of Λ , we can choose U' to contain R_0 .

The second proposition will enable us to get, starting with a preretraction $U \rightarrow G$, a preretraction from U to a subgroup G' over which G has a structure of hyperbolic tower. It appears as Proposition 5.12 in [Per09].

Proposition 6.8: *Let G be a torsion-free hyperbolic group. Let U be a non cyclic retract of G , and let Λ be a cyclic (relative) JSJ decomposition of U . Suppose G' is a subgroup of G containing U such that either G' is a free factor of G , or G' is a retract of G by a retraction $r : G \rightarrow G'$ which makes (G, G', r) a hyperbolic floor. If there exists a non-injective preretraction $U \rightarrow G$ with respect to Λ , then there exists a non-injective preretraction $U \rightarrow G'$ with respect to Λ .*

7 Homogeneity of the free groups

The aim of this section is to show that non abelian free groups are strongly \aleph_0 -homogeneous. We start by showing that finitely generated free groups are homogeneous.

7.1 Finitely generated free groups

Let G and G' be groups, and let B and B' be subgroups of G and G' respectively which are isomorphic via $f : B \rightarrow B'$. If ϕ is a first-order formula with parameters in B , we denote by $f(\phi)$ the formula with parameters in B' obtained from ϕ by replacing each parameter b by $f(b)$. If p is a type over B , we denote by $f(p)$ the set of formulas with parameters in B' consisting of the formulas $f(\phi)$ for $\phi \in p$.

The following result is the key intermediate step to prove homogeneity of finitely generated free groups.

Proposition 7.1: *Let G and G' be torsion-free hyperbolic groups. Let B and B' be subgroups of G and G' respectively which are isomorphic via a map $f : B \rightarrow B'$.*

Let $\bar{u} = (u_1, \dots, u_n)$ and $\bar{v} = (v_1, \dots, v_n)$ be tuples of elements of G and G' respectively. Let H_u be the subgroup generated by \bar{u} and B , and let U be a finitely presented subgroup of G which contains H_u , and is freely indecomposable with respect to it.

Suppose that $f(\text{tp}^G(\bar{u}/B)) = \text{tp}^{G'}(\bar{v}/B')$. Then either there exists an embedding $U \hookrightarrow G'$ extending f which sends \bar{u} to \bar{v} , or there exists a non injective preretraction $r : U \rightarrow G$ with respect to a JSJ decomposition Λ of U in which H_u is elliptic.

In fact, we will deduce this from the following special case where B is empty:

Proposition 7.2: *Let G and G' be torsion-free hyperbolic groups. Let $\bar{u} = (u_1, \dots, u_n)$ and $\bar{v} = (v_1, \dots, v_n)$ be tuples of elements of G and G' respectively. Let U be a finitely presented subgroup of G which contains $H_u = \langle \bar{u} \rangle$, and is freely indecomposable with respect to it.*

Suppose that $\text{tp}^G(\bar{u}) = \text{tp}^{G'}(\bar{v})$. Then either there exists an embedding $U \hookrightarrow G'$ which sends \bar{u} to \bar{v} , or there exists a non-injective preretraction $r : U \rightarrow G$ with respect to the JSJ decomposition Λ of U relative to H_u .

Let us first see why this implies the more general result stated in Proposition 7.1.

Proof of Proposition 7.1. We also denote by f the extension of f to H_u which sends \bar{u} to \bar{v} .

By Lemma 3.8, there is a finitely generated subgroup H_0 of H_u such that U is freely indecomposable with respect to H_0 , H_u is elliptic in the JSJ decomposition Λ of U with respect to H_0 , and $\text{Mod}_{H_0}(U) = \text{Mod}_{H_u}(U)$. Moreover, by Corollary 4.5, we may also assume H_0 large enough so that an embedding $U \hookrightarrow G'$ which restricts to f on H_0 also restricts to f on H_u .

Now if we let \bar{h} be a finite generating set for H_0 , it is easy to see that $\text{tp}^G(\bar{h}) = \text{tp}^{G'}(f(\bar{h}))$, so by Proposition 7.2, either there is an embedding $U \hookrightarrow G'$ restricting to f on H_0 , or there is a non injective preretraction $U \hookrightarrow G$ with respect to Λ . This implies the result. \square

To prove Proposition 7.2, we need the following definition.

Definition 7.3: Let U be a group, let Λ be a (relative) cyclic JSJ decomposition of U , and let h be a morphism from U to a group G . We say that a morphism $h' : U \rightarrow G$ is Λ -related to h if

- for each rigid vertex group R of Λ , there exists an element u_R such that the restriction of h' to R is $\text{Conj}(u_R) \circ h|_R$;
- for each flexible vertex group S of Λ , if S has non abelian image by h , it also has non abelian image by h' .

Remark 7.4: Suppose U is a subgroup of a group G , and let Λ be a cyclic (relative) JSJ decomposition for U .

- h is Λ -related to an embedding $U \hookrightarrow G$ iff it is a preretraction with respect to Λ ;
- if σ is a (relative) modular automorphism, then by Lemma 3.5, the map $h' = h \circ \sigma$ is Λ -related to h .

The following straightforward lemma, which states that Λ -relatedness can be expressed in first-order logic, is stated as Lemma 5.17 in [Per09].

Lemma 7.5: Let U be a group generated by a finite tuple \bar{u} , and let Λ be a (relative) cyclic JSJ decomposition of U . There exists a first order formula $\text{Rel}(\bar{x}, \bar{y})$ such that for any pair of morphisms h and h' from U to G , the morphism h' is Λ -related to h if and only if $G \models \text{Rel}(h(\bar{u}), h'(\bar{u}))$.

Proof of Proposition 7.2. Denote by f the homomorphism $H_u \rightarrow G'$ sending \bar{u} to \bar{v} .

Suppose that there is no embedding $U \hookrightarrow G'$ extending f . By Proposition 4.4, there exists a finite set of proper quotients of U through which any morphism $U \rightarrow G'$ extending f factors after precomposition by an element σ of the modular group $\text{Mod}_{H_u}(U)$.

Pick a non trivial element g_j in the kernel of each of these quotients $\eta_j : U \rightarrow U_j$. Pick a finite presentation $\langle \bar{s} \mid \Sigma_U(\bar{s}) \rangle$ for U . The elements of the tuple \bar{u} and the elements g_j are represented by words $\bar{u}(\bar{s})$ and $g_j(\bar{s})$ in \bar{s} .

For any morphism $\theta : U \rightarrow G'$ which extends f , there exists a morphism $\theta' : U \rightarrow G'$ (namely the morphism $\theta \circ \sigma$) which is Λ -related to it by Remark 7.4, and for which $\theta'(g_j) = 1$ for some j . We can express this by a first order sentence $\phi(\bar{v})$ as follows:

$$\forall \bar{x} \left\{ [\Sigma_U(\bar{x}) = 1 \wedge (\bar{v} = \bar{u}(\bar{x}))] \Rightarrow \exists \bar{y} [\Sigma_U(\bar{y}) = 1 \wedge \text{Rel}(\bar{x}, \bar{y}) \wedge \bigvee_j g_j(\bar{y}) = 1] \right\}.$$

We have $G' \models \phi(\bar{v})$, so by hypothesis $G \models \phi(\bar{u})$. The interpretation of $\phi(\bar{u})$ gives us that for any morphism $\theta : U \rightarrow G$ which fixes \bar{u} , there is a non injective morphism $\theta' : U \rightarrow G$ which is Λ -related to θ . If we take θ to be the embedding of U in G , by Remark 7.4 we get a non injective preretraction $U \rightarrow G$ with respect to Λ . \square

In the particular case where $G = G'$, Proposition 7.1 implies

Theorem 7.6: *Let G be a torsion-free hyperbolic group, and let B be a subgroup of G . Let $\bar{u} = (u_1, \dots, u_n)$ and $\bar{v} = (v_1, \dots, v_n)$ be tuples of elements of G .*

Suppose that $\text{tp}^G(\bar{u}/B) = \text{tp}^G(\bar{v}/B)$. Then either there is an automorphism of G which restricts to the identity on B and sends \bar{u} to \bar{v} , or there exists a retraction r on G which makes $(G, r(G), r)$ a hyperbolic floor, and such that either $H_u = \langle B, \bar{u} \rangle$ or $H_v = \langle B, \bar{v} \rangle$ lies in $r(G)$.

Proof. Denote by U (respectively V) the smallest free factor of G containing H_u (respectively H_v). Any injective morphism $i : U \rightarrow G$ restricting to the identity on B and sending \bar{u} to \bar{v} has image in V , and conversely any injective morphism $j : V \rightarrow G$ restricting to the identity on B and sending \bar{v} to \bar{u} has image in U .

Suppose first that such morphisms i, j exist: then the morphism $j \circ i$ is an injective morphism $U \rightarrow U$ which fixes H_u . By Theorem 4.2, it is in fact an automorphism. Thus i is an isomorphism $U \rightarrow V$. By Grushko's theorem, it can be extended to an isomorphism $G \rightarrow G$ since U and V are free factors of G . Since this map sends \bar{u} to \bar{v} and fixes B , the first alternative is satisfied.

But if (without loss of generality) there is no injective morphism $U \rightarrow G$ fixing B and sending \bar{u} to \bar{v} , we can apply Theorem 7.1 to get a non injective preretraction $U \rightarrow G$ with respect to the JSJ decomposition of U relative to H_u . Note also that in this case, U is not cyclic: if U was cyclic generated by an element u_0 , all the elements of \bar{u} and of B would be powers of u_0 , thus by equality of the types all the elements of \bar{v} and of B would be powers of a common element v_0 . Then the map $U \rightarrow G$ sending u_0 to v_0 is clearly injective, fixes B , and sends \bar{u} to \bar{v} .

By Proposition 6.8, there exists a non injective preretraction $U \rightarrow U$. By Proposition 6.7, we get a retraction $r : U \rightarrow r(U)$ which makes $(U, r(U), r)$ into a hyperbolic floor, and we can choose it so that the vertex group of Λ containing H_u lies in $r(U)$. This easily extends to a hyperbolic floor structure of G over a proper subgroup containing H_u . \square

Corollary 7.7: *Let \bar{u} and \bar{v} be finite tuples of elements of a finitely generated free group \mathbb{F} , and let B be a subgroup of \mathbb{F} . If $\text{tp}^{\mathbb{F}}(\bar{u}/B) = \text{tp}^{\mathbb{F}}(\bar{v}/B)$, then there exists an automorphism of \mathbb{F} which restricts to the identity on B and sends \bar{u} to \bar{v} .*

Proof. If there is no automorphism of \mathbb{F} sending \bar{u} to \bar{v} and fixing B , then Theorem 7.6 implies that \mathbb{F} admits a structure of hyperbolic floor over a proper subgroup. But by Proposition 6.5, no such structure exists. \square

Remark 7.8: *Similarly, the fundamental groups of the sum of three projective planes and four projective planes respectively are homogeneous, since such groups do not admit proper hyperbolic tower structures.*

7.2 Free groups of infinite rank

We now show that any free group of infinite rank is strongly \aleph_0 -homogeneous. We begin by recalling the proof of the following classical fact of model theory (for a definition of elementary chain, see Section 2):

Lemma 7.9: *The union of an elementary chain of \aleph_0 -homogeneous structures is \aleph_0 -homogeneous.*

Proof. Let $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$ be the union of an elementary chain of \aleph_0 -homogeneous structures. Let \bar{a}, \bar{b} be finite tuples of \mathcal{M} , with $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b})$, and let c be an element of \mathcal{M} . There is an index $j \in I$ such that \bar{a}, \bar{b}, c are in \mathcal{M}_j . By the elementary chain theorem mentioned in Section 2, \mathcal{M}_j is an elementary substructure of \mathcal{M} , thus $tp^{\mathcal{M}_j}(\bar{a}) = tp^{\mathcal{M}_j}(\bar{b})$. Now \mathcal{M}_j is \aleph_0 -homogeneous so there is $d \in \mathcal{M}_j$, such that $tp^{\mathcal{M}_j}(\bar{a}c) = tp^{\mathcal{M}_j}(\bar{b}d)$. This implies $tp^{\mathcal{M}}(\bar{a}c) = tp^{\mathcal{M}}(\bar{b}d)$. \square

Proposition 7.10: *Let $\kappa \geq \aleph_0$. Then \mathbb{F}_κ is strongly \aleph_0 -homogeneous.*

Proof. As noted in Section 2, \mathbb{F}_{\aleph_0} is the union of the elementary chain $(\mathbb{F}_n)_{n \in \omega}$. By Theorem 7.1, this is in fact an elementary chain of \aleph_0 -homogeneous structures, so by Lemma 7.9, \mathbb{F}_{\aleph_0} is \aleph_0 -homogeneous. We pick an injection $\aleph_0 \hookrightarrow \kappa$, and see \mathbb{F}_{\aleph_0} as a free factor of \mathbb{F}_κ via the induced natural embedding. Recall that by Lemma 2.12, this embedding is elementary in \mathbb{F}_κ . Now let \bar{a}, \bar{b} be finite tuples of \mathbb{F}_κ , with $tp^{\mathbb{F}_\kappa}(\bar{a}) = tp^{\mathbb{F}_\kappa}(\bar{b})$. As \bar{a} is a finite tuple, we can find $f_1 \in \text{Aut}(\mathbb{F}_\kappa)$ such that $f_1(\bar{a}) = \bar{a}' \subset \mathbb{F}_{\aleph_0}$. Similarly, we can find $f_2 \in \text{Aut}(\mathbb{F})$ such that $f_2(\bar{b}) = \bar{b}' \subset \mathbb{F}_{\aleph_0}$. Since \mathbb{F}_{\aleph_0} is an elementary substructure, we have $tp^{\mathbb{F}_{\aleph_0}}(\bar{a}') = tp^{\mathbb{F}_{\aleph_0}}(\bar{b}')$. But \mathbb{F}_{\aleph_0} is \aleph_0 -homogeneous and countable, therefore it is strongly \aleph_0 -homogeneous. Hence, there is f in $\text{Aut}(\mathbb{F}_{\aleph_0})$ such that $f(\bar{a}') = \bar{b}'$. As \mathbb{F}_{\aleph_0} is a free factor of \mathbb{F}_κ , there is $f' \in \text{Aut}(\mathbb{F}_\kappa)$ extending f . Finally, $f_2^{-1} \circ f' \circ f_1$ is an automorphism of \mathbb{F}_κ which sends \bar{a} to \bar{b} . \square

8 Characterization of elements of primitive type

We want to understand better the type p_0 of a primitive element in a free group. The aim of this section is to give a characterization of elements which realize this type in a finitely generated group G . Note that if such an element exists, it implies in particular that G is elementary equivalent to a non abelian free group (the type of an element of G contains in particular all the closed formulas which are true on G). Thus, as noted in Remark 6.3, the group G must be torsion-free hyperbolic.

Let \mathbb{F}_n be the free group on a_1, \dots, a_n for $n \geq 2$. We show the following result

Proposition 8.1: *Let G be a finitely generated group. Let (u_1, \dots, u_k) be a k -tuple of elements of G for $1 \leq k \leq n$ such that $tp^G(u_1, \dots, u_k) = tp^{\mathbb{F}_n}(a_1, \dots, a_k)$. Then the subgroup $H_u = \langle u_1, \dots, u_k \rangle$ is free of rank k , and G admits a structure of hyperbolic tower over H_u .*

Remark 8.2: *The fundamental group of the connected sum of four projective planes does not admit any non trivial structure of hyperbolic tower, so it admits no structure of hyperbolic tower over a cyclic subgroup. This implies that the generic type is not realised in this model of the theory of free groups.*

To prove Proposition 8.1, we need the following definition:

Definition 8.3: A hyperbolic subtower of a finitely generated group G is a subgroup G' of G such that G has a structure of hyperbolic tower over G' . If G_1 and G_2 are hyperbolic subtowers of G , we say G_2 is smaller than G_1 if G_1 admits a structure of hyperbolic tower over G_2 (so in particular G_2 is a subgroup of G_1).

Lemma 8.4: If G is torsion-free hyperbolic, for any subset A of G there exists a minimal hyperbolic subtower containing A .

Note that the minimal hyperbolic subtower containing $\{1\}$ is precisely Sela's elementary core, defined in [Selb, Definition 7.5].

Proof. Any subtower of G is finitely generated, since it is a retract of G . As a finitely generated subgroup of G , it is a G -limit group. By Theorem 1.12 in [Selb], an infinite sequence of epimorphisms between G -limit groups stabilizes. \square

Proof of Proposition 8.1. It is easy to see that H_u is free of rank k .

Let U be a minimal subtower of G containing H_u . If $U = H_u$, the result holds, so let us assume this is not the case. Note that, as a minimal subtower containing H_u , the group U is freely indecomposable with respect to H_u . Suppose j is an injective morphism $U \rightarrow \mathbb{F}$ sending u_i to a_i : it must send U into $\langle a_1, \dots, a_k \rangle$ since this is the minimal free factor of \mathbb{F} containing $j(H_u)$. But $j(H_u) = \langle a_1, \dots, a_k \rangle$ which contradicts injectivity of j . Thus there are no injective morphisms $j : U \rightarrow \mathbb{F}$ sending u_i to a_i .

As noted above, G is torsion-free hyperbolic, so we can apply Theorem 7.1. Thus there exists a non-injective preretraction $U \rightarrow G$ with respect to a JSJ decomposition Λ of U in which H_u is elliptic. By Proposition 6.8, since G is a hyperbolic tower over U , there exists a non injective preretraction $U \rightarrow U$ with respect to Λ . Finally, applying Proposition 6.7, we get that there is a retraction r from U to a proper subgroup U' containing H_u such that (U, U', r) is a hyperbolic floor. This contradicts the minimality of U . \square

In fact, we will show that the converse of Theorem 8.1 holds, so that we have

Theorem 8.5: Let G be a finitely generated group. Let (u_1, \dots, u_k) be a k -tuple of elements of G for $k \leq n$. Then $\text{tp}^G(u_1, \dots, u_k) = \text{tp}^{\mathbb{F}^n}(a_1, \dots, a_k)$ if and only if the subgroup $H_u = \langle u_1, \dots, u_k \rangle$ is free of rank k and G admits a structure of hyperbolic tower over U .

For this, we will need to use

Theorem 8.6 ([Sela]): Let G be a torsion-free hyperbolic group. If G has a structure of hyperbolic tower over a non abelian subgroup H , then H is elementarily embedded in G .

The proof of this result has not appeared in print, but follows exactly the proof of Theorem 7.6 in [Selb], where Sela shows that in a torsion-free hyperbolic group G which is not elementary equivalent to the free group, the embedding of a minimal hyperbolic subtower over $\{1\}$ (an "elementary core" of G) is elementary in G . The hypothesis that G is not elementary equivalent to the free group only serves to deduce that such a minimal hyperbolic subtower is not trivial, and the minimality hypothesis is not used.

Proof of Theorem 8.5. Proposition 8.1 proves one direction of this equivalence. To see the other, we suppose first that $k \geq 2$. The group $H_u = \langle u_1, \dots, u_k \rangle$ is not abelian, so by Theorem 8.6, it is elementarily embedded in G . Since it is free of rank k , we have $\text{tp}^U(u_1, \dots, u_k) = \text{tp}^{\mathbb{F}_k}(a_1, \dots, a_k)$ where \mathbb{F}_k is the free factor of \mathbb{F}_n generated by a_1, \dots, a_k . Now by Theorem 8.6

again, this free factor is elementary in \mathbb{F}_n so $\text{tp}^{\mathbb{F}_k}(a_1, \dots, a_k) = \text{tp}^{\mathbb{F}_n}(a_1, \dots, a_k)$. This proves the result for $k \geq 2$.

Suppose now $k = 1$. Let G_0 be the bottom floor of the hyperbolic tower structure G admits over $\langle u \rangle$. Note that G_0 is in particular non abelian. By Theorem 8.6, the embedding of G_0 in G is elementary, so it is enough to show that $\text{tp}^{G_0}(u) = \text{tp}^{\mathbb{F}_n}(a_1)$. We assume without loss of generality that G_0 is minimal in the following sense: it does not admit a hyperbolic floor structure over a proper subgroup G'_0 containing u .

Now G_0 is the free product of $\langle u \rangle$ with possibly some closed surface groups of characteristic at most -2 , and possibly a free group. Note that if a closed surface group of characteristic at most -2 is not the fundamental group of the connected sum of four projective planes, it has a non-trivial structure of hyperbolic tower. On the other hand, if $S = \langle a, b, c, d \mid a^2b^2 = c^2d^2 \rangle$ is the fundamental group of the connected sum of four projective planes, then the morphism r from $\langle u \rangle * S$ to $\langle u \rangle * \langle a \rangle$ given by $r(u) = u$, $r(a) = a$, $r(c) = a$ and $r(b) = r(d) = u$ is a retraction which makes $(\langle u \rangle * S, \langle u, a \rangle, r)$ into a hyperbolic floor. Thus no closed surface group can appear in the free product decomposition of G_0 as it would contradict its minimality.

But now, G_0 has a free factor of the form $\langle u \rangle * \mathbb{Z}$, which by Theorem 8.6 is elementarily embedded in G_0 (and thus in G). Thus the type of u in G_0 is the same as the type of u in $\langle u \rangle * \mathbb{Z}$, which is precisely $\text{tp}^{\mathbb{F}_n}(a_1)$. \square

9 Surface groups

Let us now consider the fundamental group S of a closed hyperbolic surface Σ which is not the connected sum of three projective planes. By Theorem 6.4, the group S is elementary equivalent to the non abelian free group \mathbb{F} over a_1, \dots, a_n . We want to find elements whose type in S is the same as the type p_0 of a_1 in \mathbb{F} . We already remarked that there are no such elements if S is the connected sum of four connected planes, we can thus exclude this case as well.

Theorem 8.5 tells us S must have a structure of hyperbolic tower over the subgroup generated by such an element. Let us recall the description of the non trivial hyperbolic towers structures S admits given in [Per09]. Minor subsurfaces are defined by

Definition 9.1: Let Σ be a closed surface. Let Σ_0 be a proper connected subsurface of Σ . Denote by Σ_1 the closure of $\Sigma \setminus \Sigma_0$. We say that Σ_0 is a minor subsurface of Σ if

1. Σ_1 is connected;
2. $\chi(\Sigma_0) \geq \chi(\Sigma)/2$, with equality if and only if Σ_0 and Σ_1 are homeomorphic;
3. if Σ_0 is non orientable, so is Σ_1 .

The following is stated as Theorem 5.20 in [Per09]:

Proposition 9.2: Let S be the fundamental group of a closed hyperbolic surface Σ which is not the connected sum of three or four projective planes. Then S admits a structure of hyperbolic tower over a proper subgroup H if and only if H is a free factor of the fundamental group of a minor subsurface of Σ .

In particular, if S_0 is the fundamental group of a minor subsurface of Σ , by Theorem 8.6 it is an elementary subgroup of S and $\text{tp}^{S_0}(u) = \text{tp}^S(u)$ for any element u of S_0 . Thus any primitive element u of the free group S_0 is such that $\text{tp}^S(u) = \text{tp}^{\mathbb{F}}(a_1)$.

If u is an element of S which represents a non separating simple closed curve γ on Σ , it is easy to see that we can find a minor subsurface Σ_0 of Σ containing γ such that u is primitive in $S_0 = \pi_1(\Sigma_0)$. Thus elements representing non separating simple closed curves on Σ have the same type in S as primitive elements in \mathbb{F}_n .

Remark 9.3: *If u represents a separating simple closed curve, it can be written as a non trivial product of commutators and squares, so its type is not $\text{tp}^{\mathbb{F}}(a_1)$.*

If Σ is non orientable, it admits non separating simple closed curves which are 1-sided and some which are 2-sided. The elements representing each of these types of curves both have the type of a_1 in \mathbb{F} , but they are not in the same orbit under $\text{Aut}(S)$. Thus, we see that non orientable surfaces are not homogeneous. To see this for all surfaces, we will have to consider also elements which do not represent simple closed curves. Denote by $\mathcal{C}(\Sigma)$ the set of elements of S which represent simple closed curves of Σ .

Proposition 9.4: *Let S be the fundamental group of a closed hyperbolic surface Σ which is not the connected sum of three or four projective planes. There exists an element u of $S - \mathcal{C}(\Sigma)$ such that $\text{tp}^S(u) = \text{tp}^{\mathbb{F}}(a_1)$.*

Since $\text{Aut}(S)$ preserves $\mathcal{C}(\Sigma)$, Proposition 9.4 implies

Corollary 9.5: *Let S be the fundamental group of a closed hyperbolic surface Σ which is not the connected sum of four projective planes. Then S is not homogeneous.*

To prove Proposition 9.4, we need the following result. It appears as Remark 7.2 in [BS85] (see also [Riv01] and [Mir08]).

Theorem 9.6: *Let Σ be a closed hyperbolic surface. There exists a polynomial P_{Σ} such that the number of closed simple geodesic curves on Σ of length at most L is bounded above by $P_{\Sigma}(L)$.*

From this we get:

Corollary 9.7: *Let S be the fundamental group of a closed hyperbolic surface Σ . Fix a finite generating set for S , denote by B_n the set of elements of S represented by words of length at most n . Then the growth function $f_{\mathcal{C}(\Sigma)} : \mathbb{N} \rightarrow \mathbb{N}$ given by $f_{\mathcal{C}(\Sigma)}(n) = |B_n \cap \mathcal{C}(\Sigma)|$ is bounded by a polynomial in n .*

This follows from Theorem 9.6 using the fact that the isotopy class of a simple closed curve on Σ contains exactly one geodesic representative, and the S -equivariant quasi-isometry between the Cayley graph of S and the universal cover of Σ .

Proof of Proposition 9.4. Let S_0 be the fundamental group of a minor subsurface of Σ . We saw that any primitive element u of S_0 is such that $\text{tp}^S(u) = \text{tp}^{\mathbb{F}}(a_1)$. But the set of primitive elements of S_0 grows exponentially in S_0 , and thus in S as well. The set $\mathcal{C}(\Sigma)$ grows at most polynomially in S by Theorem 9.7. This proves the result. \square

Note that Proposition 9.4 also implies:

Corollary 9.8: *Let S be the fundamental group of a closed hyperbolic surface Σ which is not the connected sum of three or four projective planes. The set $\mathcal{C}(\Sigma)$ is not definable over the empty set.*

We showed that two elements representing non separating simple closed curves on a surface have the same type, and they are in the same orbit if and only if the corresponding curves

are both one-sided, or both two-sided. To complete the picture regarding the types of simple closed curves, we prove

Proposition 9.9: *Let S be the fundamental group of a closed hyperbolic surface Σ which is not the connected sum of three or four projective planes. Elements representing separating simple closed curves have the same type if and only if they are in the same orbit under $\text{Aut}(S)$.*

Proof. Let c and d be two elements of S representing separating simple closed curves γ and δ on Σ . Suppose that $\text{tp}^S(c) = \text{tp}^S(d)$. Denote by Σ_c^1 and Σ_c^2 (respectively Σ_d^1 and Σ_d^2) the connected components of $\Sigma - \gamma$ (respectively $\Sigma - \delta$). We denote the fundamental group of Σ_ϵ^i by S_ϵ^i for $i \in \{1, 2\}$ and $\epsilon \in \{c, d\}$. We define the complexity of a surface with boundary Θ to be $k(\Theta) = (r, -n)$, where r is the rank of its fundamental group as a free group and n is the number of its boundary components, and we order complexities lexicographically.

Assume without loss of generality that $k(\Sigma_c^1) = \min_{i,\epsilon} \{k(\Sigma_\epsilon^i)\}$. If Σ_c^1 is orientable, S_c^1 admits a presentation of the form $\langle c, a_1, \dots, a_{2r} \mid c = \prod_j [a_{2j-1}, a_{2j}] \rangle$. The formula $\exists x_1, \dots, x_{2r} \ c = \prod_j [x_{2j}, x_{2j+1}]$ is in the type of c in S , hence it is also satisfied by d . This gives us a morphism $f : S_c^1 \rightarrow S$ sending c to d . We can proceed in the same way for the non orientable case.

Denote by Λ_d the graph of group decomposition of S dual to δ , it corresponds to an amalgamated product of the form $S_d^1 *_{\langle d \rangle} S_d^2$. Note that since the centralizer of d in S is exactly $\langle d \rangle$, the tree corresponding to Λ_d is 1-acylindrical, that is, distinct edges have stabilizers which intersect trivially.

Suppose first that the kernel of the map f does not contain any elements which represent simple closed curves on Σ_c^1 . The surface group with boundary S_c^1 acts on the tree corresponding to Λ_d via the map f , and boundary elements of S_c^1 are elliptic in this action. By Theorem III.2.6 of [MS84], the splitting induced on S_c^1 by this action is dual to a non empty set of disjoint simple closed curves on Σ_c^1 , and its vertex groups are fundamental groups of proper subsurfaces of Σ_c^1 . Denote by \hat{S}_c^1 the fundamental group of one of these subsurface $\hat{\Sigma}_c^1$, and suppose without loss of generality that $f(\hat{S}_c^1) \leq S_d^1$. The map f sends boundary elements of \hat{S}_c^1 to boundary elements of S_d^1 , so by Lemma 3.10 in [Per09], $f(\hat{S}_c^1)$ has finite index in S_d^1 . By Lemma 3.12 in [Per09], this implies that the complexity of $\hat{\Sigma}_c^1$, and thus that of Σ_c^1 , is at least that of Σ_d^1 , and if we have equality f is an isomorphism $S_c^1 \rightarrow S_d^1$. By minimality of $k(\Sigma_c^1)$, we do have equality.

If the kernel of the map f does contain elements representing simple closed curve on Σ_c^1 , we proceed as in Section 5.7 of [Per09]. We pick an essential set of curves killed by f , and define the pinching map ρ_C and the graph of groups $\Gamma(S_c^1, C)$. The map f then factors as $f' \circ \rho_C$, and the kernel of f' does not contain elements which represent simple closed curves on the surfaces of $\Gamma(S_c^1, C)$, including on the unique surface $\hat{\Sigma}_c^1$ whose fundamental group contains $\rho_C(d)$. By repeating the argument above for the map f' and the surface $\hat{\Sigma}_c^1$, we see that the complexity of $\hat{\Sigma}_c^1$ is at least that of Σ_d^1 . But $k(\hat{\Sigma}_c^1) < k(\Sigma_c^1)$, this is a contradiction.

Thus we have an isomorphism $f : S_c^1 \rightarrow S_d^1$ sending c to d . To finish the proof, it is enough to find an isomorphism $h : S_c^2 \rightarrow S_d^2$ sending c to d . Since $\chi(\Sigma_c^1) + \chi(\Sigma_c^2) = \chi(\Sigma) = \chi(\Sigma_d^1) + \chi(\Sigma_d^2)$, and all the surfaces involved have one boundary component, we see that we must have $k(\Sigma_c^2) = k(\Sigma_d^2)$. If Σ_c^2 and Σ_d^2 have the same orientability, we are done. Thus let us assume that Σ_c^2 is orientable and Σ_c^2 isn't. In this case, note that Σ is non orientable so Σ_c^1 and Σ_d^1 must be non orientable.

As we did for S_c^1 , we can find (using equality of the types of c and d) a morphism $h : S_d^2 \rightarrow S$ sending c to d . Again we start by assuming no elements representing simple closed curves on

Σ_c^2 are in the kernel of h . If $h(S_c^2)$ is not elliptic, by acylindricity of Λ_d we can find a proper subsurface of Σ_c^2 whose fundamental group is sent by h to a conjugate of S_d^2 . But then its complexity must be at least that of Σ_d^2 , while being strictly smaller than that of Σ_c^2 , this is a contradiction. Thus $h(S_c^2)$ is elliptic in Λ_d , i.e. $h(S_c^2) \leq S_d^j$ for $j \in \{1, 2\}$. If $\text{Ker}(h)$ does contain elements representing simple closed curves, as before we can apply this argument to a surface obtained by pinching to get a contradiction.

The morphism h sends non conjugate maximal boundary elements of S_c^2 to non conjugate maximal boundary elements of S_d^j , so by Lemma 5.22 of [Per09], it is an isomorphism. But both Σ_d^1 and Σ_d^2 are non orientable, which contradicts orientability of Σ_c^2 . \square

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